FURTHER MATHEMATICS

Paper 9795/01
Further Pure Mathematics

Key messages

Candidates need to take note of the structure of questions and look for links between the different parts of questions in order to ensure that they do not waste time doing unecessary or repetative work.

General Comments

There were some excellent scripts seen with candidates scoring over 100, a commendable achievement. However, there appeared to be three major difficulties arising and contributing to an overall depression of the totals gained on this year's paper. Firstly, many candidates spent too long on trivial matters rather than getting straight down to business with the real mathematics involved in some of the questions. Secondly, it was quite noticeable that many candidates failed to perceive the helpfulness of the paper's setters, and many candidates frequently went on to "undo" the helpful structuring of questions that was intended to guide and help them, proceeding effectively to start from scratch: this merely wasted time and effort, and often led them to do unnecessary work or work leading in another direction altogether. **Questions 1, 10, 11, 12 and 13** were particular targets of inefficiency in this respect. Finally, **Question 9** was found universally difficult with very few candidates able to deal with explanatory question on the Groups topic.

For several of the reasons mentioned already, the paper as a whole proved slightly on the long side, and **Question 13** also scored very poorly for candidates, partly because of its unstructured nature but mostly because many candidates appeared to lack the time to make a good attempt at it.

Comments on specific questions

Question 1

This was intended to be a simple starter. However, many candidates appeared to be unable to write out the required six terms correctly in part **(i)**. The real purpose of this part was to highlight that there were actually $(N + 1)$ terms in the summation, and to guide candidates to do part **(ii)** by subtracting the sum of the first $(N-1)$ squares from the sum of the first $(2N)$ squares – also firmly hinted at in the wording of part **(ii)**. In the event, most candidates did part **(ii)** by expanding the square and splitting the summation into three parts. This is a perfectly valid approach, but the majority of these candidates then thought that $\sum_{r=0}^{N} N^2$ was $N \times N^2$ rather than $(N + 1) \times N^2$ because they had omitted to pick up on the extra term they had been directed towards in part **(i)**. It is worth stressing to candidates that questions are carefully worded to be as helpful as possible and that part questions numbered (i), (ii) … tend to be linked.

Answers: **(i)** $N^2 + (N+1)^2 + (N+2)^2 + ... + (2N-2)^2 + (2N-1)^2 + (2N)^2$; **(ii)** $a = 14$.

Question 2

This question was completed well by almost all candidates. However, it must be pointed out that when the question says, as it does in part **(iii)** here, "Use the result of part **(i)** to …" then candidates are unlikely to get much credit for alternative approaches. Where candidates have used their calculators, they should still present sufficient working to justify to Examiners that they have done the work to earn the marks.

Answers: **(i)** $t = -3$; $10\mathbf{I}$; **(ii)** $\begin{pmatrix} -5 & 0 & 5 \\ 12 & -2 & -6 \\ -9 & 4 & 7 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 8 \\ 12 \\ 22 \end{pmatrix}$; **(iii)** $x = y = 5.4,\ z = 7$.

Question 3

This was another popular and high-scoring question for candidates. The principal obstacles came for those who had not noted the proper use of signs (for example, noting that | z + 2 | needs to be seen as | z – (–2) |) and those who thought the second locus was that of a "complete" line.

Answers: **(i)** circle centre –2 (+ 0i) and radius 3; half-line from (0 +) i downwards to the right at 45°;
(ii) 1 (+0i).

Question 4

Most candidates managed this with little difficulty, beginning by using integration by parts suitably. However, quite a sizeable minority failed to express $(2x + 1)^{1.5}$ as $(2x+1)\sqrt{2x+1}$ so that they were then unable to proceed further.

Question 5

This was, by far, the most popular and highest-scoring question on the paper. The few candidates who had the signs the wrong way round in the numerator, or who omitted to square the denominator, when using the quotient rule of differentiation could still gain all following marks, and it also helped that most errors could be followed through in the sketch, provided that they did not completely alter the shape of the graph. However, candidates should be aware that when asked for a sketch, an accurate drawing on graph paper is not required, and can detract from the general shape of a graph as candidates try to fit their graph to calculated points. All that is required is the main features of the graph with a correct general shape.

Answers: **(i)** (–4, – 1)

Question 6

This was a very routine integrating factor differential equation question, which split the candidates into three camps. There were those who clearly either did not recognise it as such or who were unable to cope with this type of situation. Then, there were those who were careless in finding the correct integrating factor (often by forgetting to start the process with an un-coefficiented $\frac{dy}{dx}$ term). Fortunately, the majority cleared both hurdles comfortably and went on to gain most of the marks available. Just a few forgot to divide the "+ C " by x^2 before substituting in $x = y = 1$ to evaluate it.

Answer: $y = 2\ln x - 1 + \frac{2}{x^2}$.

Question 7

This question was not answered well by a large number of candidates. There were many candidates who gave the phrase "assume that it is true for $n = k$" or, "assume that $n = k$" without any indication in their working that they were aware of what this means or what its significance might be. In this particular case, simple yet robust statements such as "assume f(k) is divisible by 13", or "let f(k) = 13m" was expected. There were then many candidates who thought that the required divisibility result would appear when they examined f(k + 1) ± f(k) which was not the case. A little more thought was required, but there are actually several ways in which f(k) can be incorporated into the expression for f(k + 1) leaving an extra term which is clearly a multiple of 13. Finally, there is always one mark at the end for a clear and convincing explanation of the induction logic – which can be very brief if the preceding working has been well laid out – and many, otherwise fully correct solutions fell short of full marks on this question by dint of having omitted this final point-scoring step.

Question 8

This was a relatively simple vectors question, although many candidates did not find it so. The two marks for part **(i)** were usually gained, although rather too many thought it sufficient to prove that only one point on the line lay in the given plane, or just that the line was parallel to the plane. Part **(ii)** was generally poorly done. Many of the solutions presented consisted of what appeared to be a random selection of intersection workings and/or scalar or vector products, with no structure or purpose to them. Candidates would have been more likely to gain marks here if they had drawn a simple diagram to help clarify the situation.

Answers: **(ii)** $k = 7$.

Question 9

As already mentioned in the general points, very few candidates had a clear idea of what to do in this question. In part **(i)**, most thought that the result was a consequence of *Lagrange's Theorem*, supposing that there would always be an element of order k whenever k was a factor of $|G|$. In part **(ii)**, most attempts *started* with the assumption of commutativity rather than trying to end up here. In part **(iii)**, many candidates clearly did not know what was meant by a "permutations group". A few candidates produced an alternative counter-example to the one requested using the group of symmetries of the equilateral triangle, and these were given full credit for their efforts. Finally, in part **(iv)**, it was important to realise that the given subset is actually a group, and show it so, and *now* the use of *Lagrange* gives the required contradiction, since 4 does not divide into $4n + 2$; only a very few candidates realised what was going on here.

Question 10

This question produced a mix of responses, with many candidates scoring the majority of the marks. Part **(i)** of this question was routine enough, and almost all candidates worked it through suitably. Many candidates did not proceed any further with the question, and even when candidates did proceed it was usually with the suggestion that $x = \cos\theta$, showing that candidates had not managed to pick up on the guidance presented in the question. The extra factor of 2 put into the required answer to part **(i)** meant that the coefficient of $(\cos\theta)^6$ was 2^6 – perhaps the simplest and most obvious bit of number work on the paper – and observation of which made the acquisition of the next five marks almost inevitable. However, around half of all candidates failed to make this small step.

Answers: **(ii)** $x = 2\cos\theta$ for $\theta = \frac{\pi}{18}, \frac{5\pi}{18}, \frac{7\pi}{18}, \frac{11\pi}{18}, \frac{13\pi}{18}, \frac{17\pi}{18}$; **(iii)** $\frac{\sqrt{3}}{8}$.

Question 11

On the whole, candidates scored well on this question, with candidates averaging almost 10 out of the 15 marks. On this occasion, (r, θ) being written as (θ, r) was condoned throughout. Some candidates needlessly used calculus in part **(ii)**. In part **(iv)**, most candidates made good progress towards finding a Maclaurin series for either $e^{\sin\theta}$ or $e^{2\sin\theta}$, although some methods were longer than others. The simplest approach is to use $\sin\theta \approx \theta$ (a small-angle approximation), although it was most commonly the case that candidates found the series for $e^{\sin\theta}$ using calculus, and *then* squared up. Some candidates worked with a quartic approximation for $e^{2\sin\theta}$, which gave the slightly more accurate (3s.f.) answer of 0.205 to the true value. Overall, this was a good question for candidates.

Answers: **(i)** $(r, \theta) = (1, 0)$; **(ii)** Min. at $\left(e^{-1}, -\frac{\pi}{2}\right)$, max. at $(e, \frac{\pi}{2})$; **(iv)** 0.204 (or 0.205).

Question 12

Those candidates who followed the guidance of the question found this question a good source of marks, but those who did not found the question difficult. To begin with, there were many candidates who lost a mark in part **(i)(a)** because they went from tanh $x = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$ to $\dfrac{e^{2x} - 1}{e^{2x} + 1}$ without saying how. Since this was a given answers all steps needed to be justified fully. Again, candidates generally missed the helpful thrust of the question's wording, which tried to direct them towards the use of sinh 2*x* throughout the rest of the question by flagging this up in part **(i)(b)**. This is particularly important since it transpires that $\dfrac{dy}{dx} = \dfrac{1}{\sinh 2x}$ at the start of part **(ii)**, and the rest of the question then unfolds very easily thereafter. There are, of course, several valid alternative approaches – working with tanh *x* throughout works quite well, for instance, and several candidates clearly thought that this was the purpose of the hints from part **(i)**; those that pursued this path with confidence usually gained high scores with very little extra effort to that intended, as did those who used a substitution for the integral involving the *k* that we had been introduced to them. The biggest hurdle to successful progress in part **(ii)**, however, usually came from candidates not finding $\dfrac{dy}{dx}$ correctly at all or by not simplifying it into some suitable form once found.

Answers: **(i)(b)** $\dfrac{2k}{k^2 - 1}$; **(ii)** $\ln\dfrac{4}{3}$.

Question 13

Once again, the given equation had been set out in the form intended to be helpful to candidates; as it turned out, most candidates attempting this preferred instead to multiply throughout by *w* in order to get a cubic. Even so, it would all have been very manageable had candidates, who had customarily started by writing $w = x + iy$, deployed the fact that $x^2 + y^2 = 1$ at an earlier stage of proceedings. What was required was the simple process of equating real and imaginary parts to zero, eliminating *y* (preferably), and then sorting out the remaining algebra. Apart from those candidates who were clearly too pressed for time at the end of the examination, the efforts of the majority of candidates who had started the question foundered on a lack of confidence – they had usually written down lots of correct statements but then did not know what to do with them.

FURTHER MATHEMATICS

Paper 9795/02
Further Applications of Mathematics

Key messages

Candidates would benefit from learning a variety of methods for solving problems and practice at deciding which is the best method to use for any particular question.

General Comments

Some excellent scripts were seen, but there were also many candidates who found the paper challenging. Few candidates seemed confident in both statistics and mechanics, and very often candidates' understanding of the applied methods was inferior to their pure mathematical skills. Some very routine scenarios were often tackled as if they were unfamiliar, particularly in mechanics. The ability to recognise the appropriate method with which to start a question was insufficiently developed. All but the best candidates seemed to have only one method available of tackling any question, insisting on using that even when it was easy to see that it was not the best for the specific question. At a simple level this might be instanced by the use of v^2/r when $r\omega^2$ is plainly better (ω was needed in the answer) in **Question 7**. In **Question 4** some insisted on using a long way of finding E(X), and in **Question 8** the tendency to reach first for magnitude and direction of velocities, rather than components, certainly made things much harder for many candidates.

Comments on specific questions

***Section A*: Probability**

Question 1

In part **(i)**, most candidates were able to combine the variances successfully. However, in part **(ii)** the classic mistake $\text{Var}(X_1 + X_2 + Y_1 + Y_2) = 4\text{Var}(X) + 4\text{Var}(Y)$ was seen at least as often as the correct variance.

Answers: **(i)** 0.487 **(ii)** 0.943

Question 2

Many candidates could not see how to start this question. Those who did, and were able to use $\sigma \div \sqrt{n}$, were sometimes unable to handle the two tails, and a few who got as far as $\sqrt{n} = 3.92$ then took the square root instead of squaring. In part **(ii)** many candidates did not use an integer value of n.

Answers: **(i)** 16 **(ii)** 0.655

Question 3

This question was found difficult. Many candidates could obtain 8000 in part **(i)**, but in part **(ii)** the correct variance eluded many, and indeed a substantial number started with a confidence interval for n rather than p. The calculation of a confidence interval for a population proportion seemed unfamiliar to many, while some used $n = 8000$ instead of 400, suggesting a tendency to treat the method as a collection of meaningless symbols. In part **(iii)** few divided 400 by the end-points of the interval in part **(ii)**.

Answers: **(i)** 8000 **(ii)** (0.0247, 0.0753) **(iii)** (5310, 16200)

Question 4

This question was better answered, perhaps because it needed largely pure mathematical skills (and also because all the answers were given). Part **(i)** was usually correct, and provided candidates wrote down an appropriate integral in part **(ii)** they could generally obtain the correct formula. The same was true in part **(iii)**; most knew or found the correct terms in the expansions of $(1 - t)^{-1}$ and e^{kt}, although many then made very heavy weather of multiplying out the series, and it was sometimes hard to follow their working. In part **(iv)**, again, some attempted to find E(X) by differentiating the MGF rather than its series, and indeed some went back to integrating xf(x). As there was only one mark given for this part, their omission of looking for a quick way suggested poor examination technique.

Question 5

This question was well answered. The PGF of a Poisson distribution was given in the formula book, and many were able to obtain the standard results E(X) = Var(X) = λ. The correct approximations were almost always used, and the only common errors were in handling the continuity corrections.

Answers: **(ii)** 0.649 **(iii)** 0.123

Question 6

This question put a high premium on integration skills. Almost all candidates knew which integrals to attempt. In part **(i)** candidates who got as far as $4\tan^{-1}m = 0.5\pi$ could choose between evaluating the left-hand side at 0.41 and 0.42, or finding $\tan(\pi/8)$. Most could get the given answer in part **(ii)**, but the integral in part **(iii)** caused many problems. Some used the substitution $x = \tan\theta$, which works well enough, but the majority of successful solutions came from writing $\frac{x^2}{1+x^2} = 1 - \frac{1}{1+x^2}$. A large number of correct answers were seen to the last part, although weaker candidates, as usual, confused P(A | B) with P($A \cap B$).

Answers: **(iii)** $\left(\frac{4}{\pi} - 1\right) - \left(\frac{2}{\pi}\ln 2\right)^2$ or 0.0785 **(iv)** $\frac{2}{3}$

***Section B*: Mechanics**

Question 7

This routine question was often very poorly done. Many candidates attempted to resolve along RQ, rather than vertically and horizontally; if this is done, it would be necessary to involve the parallel component of $mr\omega^2$, and is certainly a lot harder. Perhaps the frequency of this error can be explained by the fact that last year's question involved motion in a *vertical* circle, where resolving radially *is* the correct method. Some attempted to resolve vertically for the whole system. Many used v^2/r and converted to ω via $v = r\omega$ only at the end, if at all.

Answers: $\omega = 1.83$, $m = 0.3$

Question 8

Again, this standard question was poorly done. Many candidates made things very much harder for themselves by working throughout in terms of magnitudes and directions, instead of x- and y-components. Those who worked in components throughout made the question look much simpler; the impression was that those who did not were attempting to answer the question entirely by rote. Signs were often handled correctly, but in this type of question it really is essential that candidates draw diagrams to explain their notation, both to the Examiner and to themselves. Weaker candidates did not realise that the components of velocity perpendicular to AB remained unchanged.

Answers: 5.83 ms^{-1}, 8.06 ms^{-1}

Question 9

Almost all candidates got $t = 75$ seconds in part **(i)**, although it was strange how many did not get it in one step by using $\frac{v+u}{2} = \frac{s}{t}$; many found the acceleration first. In part **(ii)** almost all candidates used $P = Fv$, but relatively few then used differential equations; it can be reliably assumed that the use of this formula generally leads to at least one differential equation. Many tried to use change of kinetic energy but were at a loss as to what to equate it to. Some fell back at some point on using equations of uniform acceleration.

Answers: **(i)** 75 s **(ii)** 69.2 s

Question 10

Almost all candidates could show that $\lambda = 250$. However, a substantial number of candidates seemed not to be familiar with the scenario of a vertical elastic string performing simple harmonic motion; only a minority attempted to find the new equilibrium position. Many obtained $\ddot{x} = 10 - 125x$ and then simply asserted that this was the SHM equation. Attempts to find the amplitude and the maximum speed were also often poor, and that usually meant that it was not possible to find a meaningful answer to part **(iii)(d)**. Some attempted to use change of energy but that was unnecessary in finding the amplitude and no help in finding the time. There was much inappropriate use of the formulae for uniform acceleration.

Answers: **(iii)(a)** 0.03 m **(b)** 0.335 ms^{-1} **(c)** 0.562 s **(d)** 0.206 s

Question 11

The derivation of the bounding parabola was unfamiliar to many candidates. A substantial number did not incorporate y into an equation of the form (quadratic) = 0; they often tried to use the quadratic formula to solve $gx^2\tan^2\alpha - 2V^2x\tan\alpha + gx^2 = 0$.

In part **(ii)(a)** many used $y = 0$ or $y = h$ instead of $y = -h$, and some thought that the maximum range was always obtained from $\alpha = 45°$. However, there were quite a lot of correct, or nearly correct, answers seen to part **(ii)(b)** in particular.

Answers: **(ii)(a)** $x = \sqrt{3}h$ **(b)** $\alpha = 30°$

Question 12

This question was found difficult. Many candidates did not know how to start, and did not draw a diagram that combined all the information appropriately. A common mistake was to assume that the magnitude of the relative velocity of the wind was the same in both cases. The most common successful attempts were those given as δ and α in the mark scheme, in that order.

Answer: Wind blows at 24.8 km h^{-1} from bearing 118°