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Cambridge Pre-U Certificate

FURTHER MATHEMATICS **9795/01**

Paper 1 Further Pure Mathematics **May/June 2016**

MARK SCHEME

Maximum Mark: 120

Published

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Question	Answer	Marks	Notes
1	$\sum_{r=1}^{n}\left(8r^3+r\right)\equiv 8\sum_{r=1}^{n}r^3+\sum_{r=1}^{n}r$	M1	Splitting into separate series
	$\equiv 8\times\frac{1}{4}n^2(n+1)^2+\frac{1}{2}n(n+1)$	M1 M1	Both used good factorisation attempt
	$\equiv\frac{1}{2}n(n+1)\left\{4n^2+4n+1\right\}$		
	$\equiv\frac{1}{2}n(n+1)(2n+1)^2$	A1 [4]	Legitimate **(AG)**
2	$\begin{pmatrix}6\\2\\5\end{pmatrix}\times\begin{pmatrix}-6\\1\\4\end{pmatrix}=3\begin{pmatrix}1\\-18\\6\end{pmatrix}$	M1 A1	Attempt at vector products of the d.v.s (any suitable multiple)
	Shortest Distance = $\lvert(\mathbf{b}-\mathbf{a})\bullet\hat{\mathbf{n}}\rvert$	M1	
	$=\frac{1}{19}\begin{pmatrix}10\\-2\\5\end{pmatrix}\bullet\begin{pmatrix}1\\-18\\6\end{pmatrix}=\frac{1}{19}(10+36+30)$	B1 B1	$\lvert\hat{\mathbf{n}}\rvert$ correct Sc. Prod. **ft** correct
	$=4$	A1 [6]	
	Alternative method: **M1 A1** for common normal $\mathbf{i}-18\mathbf{j}+6\mathbf{k}$ **M1 A1** for parallel planes $x-18y+6z=-55$ and -131 **M1 A1** for Sh.D formula, $\frac{\lvert 131-55\rvert}{\lvert\mathbf{n}\rvert}=\frac{76}{19}=4$		
3 (i)	$\frac{2x^2-x-1}{2x-3}=k\Rightarrow 2x^2-(2k+1)x+(3k-1)=0$	B1	**(AG)** Shown legitimately
	For non-real x, $(2k+1)^2-8(3k-1)<0$	M1	Considering discriminant (or equivalent)
	$4k^2-20k+9<0\Rightarrow(2k-1)(2k-9)<0$	M1	Solving from $\Delta<0$
	$\Rightarrow$ no curve for $\frac{1}{2}<k=y<\frac{9}{2}$	A1 [4]	**(AG)** Must be satisfactorily explained
(ii)	TPs at $y=\frac{1}{2}$ $y=\frac{9}{2}$	M1	First y (k) substituted back
	i.e. $2x^2-2x+\frac{1}{2}=0$ $2x^2-10x+\frac{25}{2}=0$	M1	Second y (k) substituted back
	$x=\frac{1}{2}$ $x=\frac{5}{2}$	A1A1 [4]	
	<u>Alternative method:</u> when $\Delta=0$, **M1** $x=$ "$-\frac{b}{2a}$" $=\frac{2k+1}{4}$ **M1** $\Rightarrow x=\frac{1}{2}$ $(y=\frac{1}{2})$ & $x=\frac{5}{2}$ $(y=\frac{9}{2})$ **A1 A1** **Note:** For finding TPs via $\frac{dy}{dx}=0$, max. **M1 A1** since qn. asks for a "deduce" method		

Question	Answer	Marks	Notes
4 (i)	Attempt at det(**M**) Det = 0 *Shown*	M1 A1 [2]	 (Or via full alternative algebraic method)
(ii)	$-x+3y+z=1$ $5x-y+2z=16$ $-x+y \quad =-2$ parametrisation attempt (or equivalent) started: e.g. set $x=\lambda$, then $y=\lambda-2$ complete attempt: $z=1+\lambda-3\lambda+6=7-2\lambda$ all correct (p.v. and d.v.) … may be in vector line eqn. form: $\mathbf{r}=\begin{pmatrix}0\\-2\\7\end{pmatrix}+\lambda\begin{pmatrix}1\\1\\-2\end{pmatrix}$ <u>Alternative method 1:</u> **B1** as above, followed by (e.g.): Finding two distinct points on the solution line; e.g. (2, 0, 3), (0, –2, 7) **M1 A1** Then eqn. of line containing these 2 points **M1 A1** possibly **ft** for line (of intersection) of 3 planes (given by the 3 eqns.) **B1** <u>Alternative method 2:</u> **B1** as above, followed by: Vector product of any two plane normals **M1A1** Finding coords. or p.v. of any pt. on line **B1** Eqn. of line using these results appropriately **B1** for line (of intersection) of 3 planes (given by the 3 eqns.) **B1**	 B1 M1 M1 A1A1 [6]	 for all three
5	Aux. Eqn. $m^2-4m+5=0$ $m=2\pm\mathrm{i}$ Comp. Fn. is $y_C=\mathrm{e}^{2x}\left(A\cos x+B\sin x\right)$ For Part. Intgl. try $y=y_p=a\,\mathrm{e}^{2x}$ Both $y'=2a\,\mathrm{e}^{2x}$ and $y''=4a\,\mathrm{e}^{2x}$ Subst[g]. into given d.e. & solving to find a : $y_p=24\mathrm{e}^{2x}$ Gen. Soln. $y=\mathrm{e}^{2x}\left(A\cos x+B\sin x+24\right)$	M1 A1 B1ft B1 B1 M1 A1 B1ft [8]	Including solving attempt $\left(4a-8a+5a\right)\mathrm{e}^{2x}=24\mathrm{e}^{2x}$ y_C+y_P provided y_C has 2 arbitrary constants and y_P has none. Also, A, B must be real here

Question	Answer	Marks	Notes
6 (i)	For f(x) = sinh x + sin x – 3x, f(2.5) = – 0.851… < 0 and f(3) = 1.159… > 0 Change-of-sign (for a continuous fn.) $\Rightarrow$ 2.5 < α < 3	M1 A1 [2]	or LHS < RHS and then LHS > RHS All correctly shown/explained
(ii)	$\sinh x + \sin x = \left(x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \frac{x^9}{9!} + \ldots\right) + \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \ldots\right)$ $= 2x + \frac{x^5}{60} + \ldots$ $2x + \frac{x^5}{60} = 3x \Rightarrow (x \neq 0)\ x^4 = 60$ $\Rightarrow \alpha \approx \sqrt[4]{60}$ (2.783 158 …)	M1 A1 B1 [3]	for use of both series (attempted) **(AG)** shown legitimately
(iii)	Using $2x + \frac{x^5}{60} + \frac{x^9}{181\,440} = 3x$ with $x \neq 0$ Solving as a quadratic in x^4 $\alpha \approx 2.769\,8$ (to 4 d.p.) [c.f. actual root 2.769 7 to 4 d.p.]	M1 M1 A1 [3]	 $x^8 + 3024x^4 - 181\,440 = 0$ from $x^4 = \sqrt{2\,467\,584} - 1512$, $x = \sqrt[4]{58.854\,5\ldots}$
7 (i)	$\lvert z^3 \rvert = 2\sqrt{2}$ arg(z^3) = $\frac{1}{4}\pi$ $\Rightarrow z = \left(\sqrt{2},\ \frac{1}{12}\pi\right)$ cube-rooting modulus; arg ÷3 Other two roots: $\left(\sqrt{2},\ \frac{3}{4}\pi\right)$ and $\left(\sqrt{2},\ \frac{17}{12}\pi\right)$	B1B1 M1M1 A1A1 [6]	 (in at least the first case)
(ii)	Equilateral Δ with vertices in approx. correct places Area = $3 \times \frac{1}{2} \times \sqrt{2} \times \sqrt{2} \sin\left(\frac{2}{3}\pi\right) = \frac{3}{2}\sqrt{3}$ Accept **awrt** 2.60 (3 s.f.) from correct working	B1 M1A1 [3]	 Give M1 for any correct area

Question	Answer	Marks	Notes
8 (i) (a)	(table G below)	M1	for mostly correct
		A1	for all correct
		[2]	

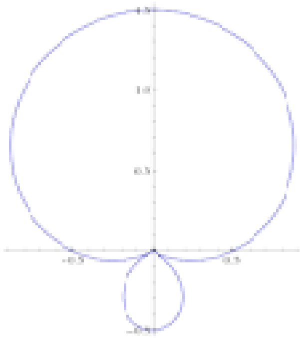
G	1	2	4	8	16	32
1	1	2	4	8	16	32
2	2	4	8	16	32	1
4	4	8	16	32	1	2
8	8	16	32	1	2	4
16	16	32	1	2	4	8
32	32	1	2	4	8	16

Question	Answer	Marks	Notes
(b)	$(S, \times_{63})$ closed, since no new elements in table	B1	
	$\times_{63}$ is associative (given)		
	1 is the identity element	B1	
	Each (non-identity) element has a unique inverse: $2 \leftrightarrow 32$, $4 \leftrightarrow 16$ and 8 is self-inverse	B1 [3]	All must be identified
(ii) (a)	(table H below)	B1	for last 3 elements (any forms)
		B1	for identity row/column (green)
		B1	for easy elements (gold) or ⩾ 14 others
		B1	for all
		[4]	

H	*e*	*x*	*y*	y^2	*xy*	*yx*
e	*e*	*x*	*y*	y^2	*xy*	*yx*
x	*x*	*e*	*xy*	*yx*	*y*	y^2
y	*y*	*yx*	y^2	*e*	*x*	*xy*
y^2	y^2	*xy*	*e*	*y*	*yx*	*x*
xy	*xy*	y^2	*yx*	*x*	*e*	*y*
yx	*yx*	*y*	*x*	*xy*	y^2	*e*

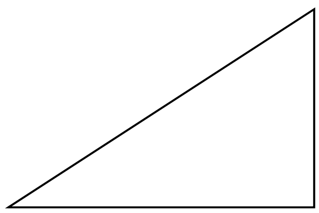
Question	Answer	Marks	Notes
(b)	Proper subgroups of *H* are (condone inclusion of $\{e\}$ and *H*): $\{e, x\}$, $\{e, xy\}$, $\{e, yx\}$ and $\{e, y, y^2\}$	B1B1 [2]	B1 Any 2; +B1 all 4 and no extras
(c)	*G* and *H* are NOT isomorphic e.g. Different numbers of self-inverse elements / elements of order 3 or *G* cyclic, *H* non-cyclic or *G* abelian, *H* non-abelian	B1 [1]	Correct conclusion WITH a valid reason

Question	Answer	Marks	Notes
9 (i)	$\alpha+\beta+\gamma=a$, $\alpha\beta+\beta\gamma+\gamma\alpha=b$ and $\alpha\beta\gamma=c$	B1B1 [2]	B1 any 2 correct; + B1 all 3 correct
(ii)	$\alpha^2+\beta^2+\gamma^2=(\alpha+\beta+\gamma)^2-2(\alpha\beta+\beta\gamma+\gamma\alpha)$	M1	
	$=a^2-2b$	A1	
	$\alpha^2\beta^2+\beta^2\gamma^2+\gamma^2\alpha^2=(\alpha\beta+\beta\gamma+\gamma\alpha)^2-2\alpha\beta\gamma(\alpha+\beta+\gamma)$	M1	
	$=b^2-2ac$	A1 [4]	
(iii)	$(\alpha-2\beta\gamma)(\beta-2\gamma\alpha)(\gamma-2\alpha\beta)$		
	$=\left(\alpha\beta-2\beta^2\gamma-2\alpha^2\gamma+4\gamma^2\alpha\beta\right)(\gamma-2\alpha\beta)$	M1	
	$=\alpha\beta\gamma-2(\alpha^2\beta^2+\beta^2\gamma^2+\gamma^2\alpha^2)+4\alpha\beta\gamma(\alpha^2+\beta^2+\gamma^2)-8(\alpha\beta\gamma)^2$	M1	Collecting up in terms of the symmetric fns.
	$=c-2\left(b^2-2ac\right)+4c\left(a^2-2b\right)-8c^2$	M1	Use of **(i)**'s and **(ii)**'s results
	$=c\left(1+4a+4a^2\right)-2\left(b^2+4bc+4c^2\right)$		
	$=c(2a+1)^2-2(b+2c)^2$	A1 [4]	**legitimately**
	<u>Alternative method:</u>		
	Using $\alpha\beta\gamma=c$, $(\alpha-2\beta\gamma)(\beta-2\gamma\alpha)(\gamma-2\alpha\beta)$		
	$=\left(\alpha-\frac{2c}{\alpha}\right)\left(\beta-\frac{2c}{\beta}\right)\left(\gamma-\frac{2c}{\gamma}\right)$		
	$=\frac{1}{\alpha\beta\gamma}\left(\alpha^2-2c\right)\left(\beta^2-2c\right)\left(\gamma^2-2c\right)=$		
	$\frac{1}{c}\left((\alpha\beta\gamma)^2-2c\sum\alpha^2\beta^2+4c^2\sum\alpha^2-8c^3\right)$		
	$=\frac{1}{c}\left(c^2-2c\left[b^2-2ac\right]+4c^2\left[a^2-2b\right]-8c^3\right)$		
	= etc. as above		
(iv)	One root is the product of the other two		
	$\Leftrightarrow\ (\alpha-2\beta\gamma)(\beta-2\gamma\alpha)(\gamma-2\alpha\beta)=0$		
	$\Leftrightarrow\ c(2a+1)^2=2(b+2c)^2$	B1	**legitimately**
	Must reason $\Rightarrow$ and $\Leftarrow$ explicitly (or together)	[1]	

Question	Answer	Marks	Notes
10 (i)		M1A1	$\frac{1}{2}+\sin\theta=0$ when $\theta=\frac{7}{6}\pi$, $\frac{11}{6}\pi$
		B1	Symmetry in y-axis
		B1	$\left(\frac{1}{2},\ 0\right)$ on initial line
		B1	Correct upper portion
		B1 [6]	Correct lower portion
(ii)	$A=\left(\frac{1}{2}\right)\int_0^{2\pi}\left(\frac{1}{2}+\sin\theta\right)^2\,\mathrm{d}\theta$	M1	Penalise incorrect multiples with final A0
	$=\frac{1}{2}\int_0^{2\pi}\left(\frac{1}{4}+\sin\theta+\frac{1}{2}-\frac{1}{2}\cos 2\theta\right)\mathrm{d}\theta$	M1	Double-angle formula
	$=\frac{1}{2}\left[\frac{3}{4}\theta-\cos\theta-\frac{1}{4}\sin 2\theta\right]_0^{2\pi}$	A1	correctly integrated 3 suitable terms
	$=\frac{3}{4}\pi$	A1 [4]	

Question	Answer	Marks	Notes
11 (i)	$F_3 = 2,\ F_4 = 3,\ F_5 = 5,\ F_6 = 8$	B1 [1]	all
(ii) (a)	$\mathrm{p}_2(x) = 1 + \dfrac{1}{x+1} = \dfrac{x+2}{x+1}$	B1	
	$\mathrm{p}_3(x) = \dfrac{2x+3}{x+2}$	B1	
	$\mathrm{p}_4(x) = \dfrac{3x+5}{2x+3}$	B1 [3]	**(AG))**
(b)	$\mathrm{p}_n(x) = \dfrac{F_n\ x + F_{n+1}}{F_{n-1}\ x + F_n}$	B1	
	Result is true for $n = 2$ (and 3 and 4)	B1	May be mentioned in later in their "round up"
	Assuming $\mathrm{p}_k(x) = \dfrac{F_k\ x + F_{k+1}}{F_{k-1}\ x + F_k}$ (not separate from their conjecture)		
	$\mathrm{p}_{k+1}(x) = 1 + \dfrac{F_{k-1}\ x + F_k}{F_k\ x + F_{k+1}}$	M1	
	$= \dfrac{F_k\ x + F_{k+1}}{F_k\ x + F_{k+1}} + \dfrac{F_{k-1}\ x + F_k}{F_k\ x + F_{k+1}}$		
	$= \dfrac{(F_k + F_{k-1})\ x + (F_k + F_{k+1})}{F_k\ x + F_{k+1}}$	M1	Collecting coeffts. into successive Fib. terms
	$= \dfrac{F_{k+1}\ x + F_{k+2}}{F_k\ x + F_{k+1}}$	A1	
	which is the required formula with $n = k + 1$. Accept this as sufficient that proof follows by induction.	[5]	

Question	Answer	Marks	Notes
12 (i)	$y=\ln\left(\tanh\frac{1}{2}x\right) \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1}{\tanh\frac{1}{2}x}\cdot\frac{1}{2}\mathrm{sech}^2\frac{1}{2}x$	M1A1	
	$=\mathrm{cosech}\,x$	A1 [3]	**(AG)**
(ii) (a)	$L_n=\int_n^{2n}\sqrt{1+\mathrm{cosech}^2x}\,\mathrm{d}x$	M1	
	$=\int_n^{2n}\coth x\,\mathrm{d}x$	A1	
	$=\left[\ln(\sinh x)\right]$	A1	correct integrn.
	$\ln\left(\frac{\sinh 2n}{\sinh n}\right)=\ln\left(\frac{e^{2n}-e^{-2n}}{e^n-e^{-n}}\right)$	M1	
	$\approx\ln\left(\frac{e^{2n}}{e^n}\right)$, for large n, $=\ln\left(e^n\right)=n$	A1 [5]	**legitimately**
	OR $\ln\left(\frac{\sinh 2n}{\sinh n}\right)=\ln\left(2\cosh n\right)=\ln\left(e^n+e^{-n}\right)$ **M1** $\approx\ln\left(e^n\right)$ for large n, $=n$ **A1**		**legitimately**
(b)	Method (sketch or statement) to indicate that C asymptotically "merges" with the x-axis	M1	
	so that C is approximately a horizontal straight-line from $(n, 0)$ to $(2n, 0)$	A1 [2]	
13 (i) (a)	Let $y=\sec^{-1}x$, i.e. $\sec y=x$ $\Rightarrow \cos y=\frac{1}{x} \Rightarrow y=\cos^{-1}\left(\frac{1}{x}\right)$	B1	
	Then $\frac{\mathrm{d}}{\mathrm{d}x}\left(\sec^{-1}x\right)=\frac{\mathrm{d}}{\mathrm{d}x}\left(\cos^{-1}\frac{1}{x}\right)$ $=-\frac{1}{\sqrt{1-(1/x)^2}}\times\frac{-1}{x^2}$	M1	(Using MF20 and the *Chain Rule*)
	$=\frac{1}{x\sqrt{x^2-1}}$	A1 [3]	**(AG)**
	[Allow **M1 A1** for valid non-"deduced" approaches]		

Question	Answer	Marks	Notes
(b)	$\int \sec^{-1} x \, . \, 1 \; dx$	M1	Use of integration by "parts"
	$= x \, . \, \sec^{-1} x - \int x \, . \, \frac{1}{x\sqrt{x^2-1}} \, dx$	A1 A1	
	$= \left[x \, . \, \sec^{-1} x - \cosh^{-1} x \right]$	A1 [4]	Condone lack of "+ C"
(ii) (a)	$\frac{1}{x\sqrt{x^2-1}} = \frac{1}{\sqrt{2}} \Rightarrow x^2\left(x^2-1\right)=2$		
	$\Rightarrow x^4 - x^2 - 2 = \left(x^2-2\right)\left(x^2+1\right) = 0$	M1	
	$\Rightarrow x = \sqrt{2}$ and $y = \tfrac{1}{4}\pi$	A1 A1	i.e. $P = (\sqrt{2}, \tfrac{1}{4}\pi)$
			
	$\frac{\tfrac{1}{4}\pi}{\sqrt{2}-c} = \frac{1}{\sqrt{2}}$	M1 A1	or by $y - \tfrac{1}{4}\pi = \frac{1}{\sqrt{2}}\left(x-\sqrt{2}\right)$ & $y=0$
	$c = \sqrt{2} - \frac{\pi\sqrt{2}}{4}$	A1 [6]	i.e. $Q = \left(\sqrt{2} - \frac{\pi\sqrt{2}}{4}, \; 0\right)$
(b)	Area $\Delta = \frac{1}{2} \times \frac{\pi\sqrt{2}}{4} \times \frac{\pi}{4} = \frac{\pi^2\sqrt{2}}{32}$	B1	
	Area under curve = $\sqrt{2} \, . \, \frac{\pi}{4} - \ln\left(1+\sqrt{2}\right)$	B1	using **(iii)**'s answer and the limits $(1, \sqrt{2})$
	Then $R = \frac{\pi^2\sqrt{2}}{32} - \frac{\pi\sqrt{2}}{4} + \ln\left(1+\sqrt{2}\right)$	M1	Difference in areas
	$= \ln\left(1+\sqrt{2}\right) - \frac{\pi(8-\pi)\sqrt{2}}{32}$	A1 [4]	**(AG)**