Cambridge Assessment International Education
Cambridge Pre-U Certificate

FURTHER MATHEMATICS **9795/01**

Paper 1 Further Pure Mathematics **May/June 2019**

MARK SCHEME

Maximum Mark: 120

Published

This mark scheme is published as an aid to teachers and candidates, to indicate the requirements of the examination. It shows the basis on which Examiners were instructed to award marks. It does not indicate the details of the discussions that took place at an Examiners' meeting before marking began, which would have considered the acceptability of alternative answers.

Mark schemes should be read in conjunction with the question paper and the Principal Examiner Report for Teachers.

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This document consists of **17** printed pages.

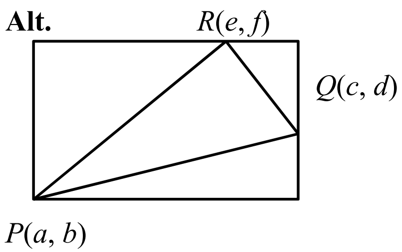
Generic Marking Principles

These general marking principles must be applied by all examiners when marking candidate answers. They should be applied alongside the specific content of the mark scheme or generic level descriptors for a question. Each question paper and mark scheme will also comply with these marking principles.

GENERIC MARKING PRINCIPLE 1: Marks must be awarded in line with: • the specific content of the mark scheme or the generic level descriptors for the question • the specific skills defined in the mark scheme or in the generic level descriptors for the question • the standard of response required by a candidate as exemplified by the standardisation scripts.
GENERIC MARKING PRINCIPLE 2: Marks awarded are always **whole marks** (not half marks, or other fractions).
GENERIC MARKING PRINCIPLE 3: Marks must be awarded **positively**: • marks are awarded for correct/valid answers, as defined in the mark scheme. However, credit is given for valid answers which go beyond the scope of the syllabus and mark scheme, referring to your Team Leader as appropriate • marks are awarded when candidates clearly demonstrate what they know and can do • marks are not deducted for errors • marks are not deducted for omissions • answers should only be judged on the quality of spelling, punctuation and grammar when these features are specifically assessed by the question as indicated by the mark scheme. The meaning, however, should be unambiguous.
GENERIC MARKING PRINCIPLE 4: Rules must be applied consistently e.g. in situations where candidates have not followed instructions or in the application of generic level descriptors.

GENERIC MARKING PRINCIPLE 5: Marks should be awarded using the full range of marks defined in the mark scheme for the question (however; the use of the full mark range may be limited according to the quality of the candidate responses seen).
GENERIC MARKING PRINCIPLE 6: Marks awarded are based solely on the requirements as defined in the mark scheme. Marks should not be awarded with grade thresholds or grade descriptors in mind.

Question	Answer	Marks	Guidance
1	For $n = 1$, LHS = $1\times2 = 2$ and RHS = $1^2\times2 = 2$	**B1**	Both sides shown equal
	Assume that the result is true for $n = k$; i.e. $\sum_{r=1}^{k} r(3r-1) = k^2(k+1)$	**M1**	Or explained later
	Then, for $n = k + 1$, $\sum_{r=1}^{k} r(3r-1) = k^2(k+1) + (k+1)(3k+2)$	**M1**	Correct $(k + 1)^{\text{th}}$ term added
	$= (k + 1)(k^2 + 3k + 2) = (k + 1)\ (k + 1)\ (k + 2)$	**M1**	Suitable factorisation algebra
	$= (k + 1)^2\ ([k + 1] + 1)$	**A1**	Shown/explained clearly
	Clear explanation that result true for $n = k \Rightarrow$ result true for $n = k + 1$ & result true for $n = 1 \Rightarrow \ldots \Rightarrow$ true $\forall\ n \in \mathbb{N}$	**A1**	Allow if BMMM earned

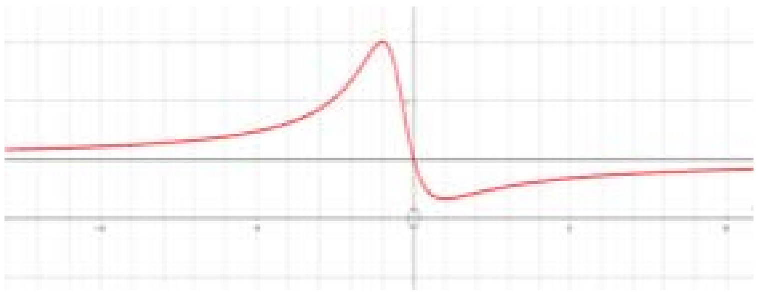
Question	Answer	Marks	Guidance
2(a)	$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{pmatrix} c-a \\ d-b \\ 0 \end{pmatrix} \times \begin{pmatrix} e-a \\ f-b \\ 0 \end{pmatrix}$ (e.g.)	**M1**	Vector product attempted
	$= \big((c-a)(f-b)-(e-a)(d-b)\big)\mathbf{k}$	**A1**	Correct, unsimplified **Must** be a vector
	Area $\Delta PQR = \frac{1}{2}\left\vert\overrightarrow{PQ} \times \overrightarrow{PR}\right\vert = \frac{1}{2}\left\vert(c-a)(f-b)-(e-a)(d-b)\right\vert$	**M1**	
	or $\frac{1}{2} \mid (ad + be + cf) - (af + bc + de) \mid$	**A1**	Correct, any form (modulus not required)
	Alt.  Area ΔPQR = Rectangle – 3 triangles **M1** $= (f-b)(c-a) - \frac{1}{2}[(d-b)(c-a) + (f-d)(c-e)$ $+ (e-a)(f-b)]$ **A1** Rectangle and $\geqslant 1\ \Delta$ **A1** All shapes = answer (any form) **A1**		*Details may be diagram-dependent, but should be consistent for full marks.*
2(b)	$\det(\mathbf{M}) = (1.c.f + 1.b.e + 1.a.d) - (1.b.c + 1.d.e + 1.a.f)$	**M1**	Any correct, unsimplified form
	Clear demonstration that this equals the bracket in **(a)**	**A1**	**Or cao** if no (viable) answer to **(a)**

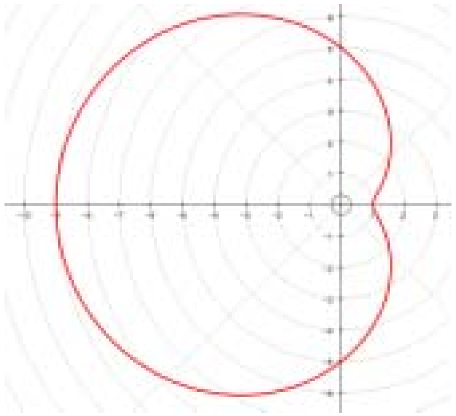
Question	Answer	Marks	Guidance
3(a)	$\begin{pmatrix}2\\3\\1\end{pmatrix}\times\begin{pmatrix}1\\-1\\2\end{pmatrix}=\begin{pmatrix}7\\-3\\-5\end{pmatrix}$	M1	
		A1	Finding a d.v.
	E.g. setting $y = 0$ & solving $2x + z = 11$ and $x + 2z = 16$ simultaneously gives the point (2, 0, 7)	M1	**Or** $(0, \frac{6}{7}, \frac{59}{7})$, $(\frac{59}{5}, -\frac{21}{5}, 0)$, etc.
		A1	
	$\mathbf{r}=\begin{pmatrix}2\\0\\7\end{pmatrix}+\lambda\begin{pmatrix}7\\-3\\-5\end{pmatrix}$ or correct equivalent	B1	**FT** Must be **r** = … Condone r not vectored.
	Alts. **I** Finding two points (with p.v.s **a** and **b**) **M1 A1 A1** Then **d** = **b** – **a** **B1 FT** and line eqn. **B1 FT** **II** Algebraic work to find d.v. (or equivalent) **M1 A1** Algebraic work to find p.v. of point on line (or equivalent) **M1 A1** Line eqn. in vector form **B1 FT**		
3(b)	Substitute $x = 2 + 7\lambda$, $y = -3\lambda$, $z = 7 - 5\lambda$ (or any individual point; e.g. (2, 0, 7)) into $4x + 11y - z = k \Rightarrow k = 1$	M1	Any correct method for find-ing a numerical value for k
		A1	**CAO**
	Alt. $3 \times ❶ - 2 \times ❷ = ❸ \Rightarrow k = 3 \times 11 - 2 \times 16 = 1$ **M1 A1**		

Question	Answer	Marks	Guidance
4(a)	$\begin{array}{c\|cccc} & 3 & 6 & 9 & 12 \\ \hline 3 & 9 & 3 & 12 & 6 \\ 6 & 3 & 6 & 9 & 12 \\ 9 & 12 & 9 & 6 & 3 \\ 12 & 6 & 12 & 3 & 9 \end{array}$		
	Multiplication Table	**M1 A1**	General idea; all correct
	Closed since no new elements in the table	**B1**	
	Identity is 6	**B1**	
	9 is self-inverse; 3, 12 an inverse-pair	**B1**	(Associativity given. Hence a group.)
4(b)(i)	Abelian? **YES** since there is symmetry about main-diagonal **Or** demonstration that $ab = ba \ \forall \ a, b$ in H **Or** $\times_{16}$ known to be commutative	**B1**	Must give a valid reason
4(b)(ii)	Cyclic? **NO** since there is no element of order 4	**B1**	Must give a valid reason
4(c)	G, H **NOT** isomorphic since G is cyclic and H isn't **or** G has elements of orders 1, 2, 4, 4 while H has elements of orders 1, 2, 2, 2	**B1**	Any correct reason

Question	Answer	Marks	Guidance
5	Aux. Eqn. $m^2 + 3m - 4 = 0$	**M1**	Solving attempt needed
	$\Rightarrow m = -4, 1$	**A1**	
	Comp. Fn. is $y = Ae^{-4x} + Be^x$	**B1**	**FT**
	For Partr. Intgl. $y = ax^2 + b + c$, $y' = 2ax + b$, $y'' = 2a$	**M1**	Including substitution into given DE
	$2a + 6ax + 3b - 4ax^2 - 4bx - 4c \equiv 1 - 8x^2$	**M1**	Comparing coeffts./terms to evaluate a, b, c
	$a = 2$, $b = 3$, $c = 3$ i.e. $y = 2x^2 + 3x + 3$	**A1**	
	Gen. Soln. is $y = Ae^{-4x} + Be^x + 2x^2 + 3x + 3$	**B1**	**FT** provided CF has 2 arb. consts. & PI none
	$y' = -4Ae^{-4x} + Be^x + 4x + 3$	**B1**	**FT** (2 expls & a quadratic)
	Substg. in for $x = 0$: $8 = A + B + 3$ and $3 = -4A + B + 3$	**M1**	Including solving attempt
	$A = 1$, $B = 4$ i.e. GS is $y = e^{-4x} + 4e^x + 2x^2 + 3x + 3$	**A1**	**CSO**
6(a)(i)	Circle	**M1**	Circle
	Centre (0, 2) and radius 3	**A1**	Details
6(a)(ii)	Line	**M1**	Line
	Perpr. bisector of line joining (0, 1) and (2, –1)	**A1**	Details
	NOTE: separate diagrams score max. 3/4		

Question	Answer	Marks	Guidance
6(b)	Circle eqn. is $x^2 + (y - 2)^2 = 9$	**B1**	Or equivalent
	Midpoint of (0, 1) and (2, –1) is (1, 0) Grad. line joining (0, 1) & (2, –1) is –1 ⇒ Perpr. gradient is 1	**M1**	Attempt at line eqn. via point and gradient
	Perpr. bisector is $y = x - 1$	**A1**	
	Alt. to line eqn. $x^2 + (y - 1)^2 = (x - 2)^2 + (y + 1)^2$ **M1** Full method $x^2 + y^2 - 2y + 1 = x^2 - 4x + 4 + y^2 + 2y + 1$ $y = x - 1$ **A1** Correct line eqn.		
6(c)	Noting that loci intersect at (0, –1) and (3, 2)	**B1**	Possibly implicitly **or** on diagram
	Area = Quarter-circle – Rt.∠d. Δ	**M1**	Must be for correct region
	$= \frac{1}{4}.9\pi - \frac{1}{2}.3.3 = \frac{9}{4}(\pi - 2)$	**A1**	Any correct, exact form
	For other region of circle, **SC B1** for $A = \frac{9}{4}(3\pi + 2)$ in place of final two marks		

Question	Answer	Marks	Guidance
7(a)	$k = \dfrac{x^2 - x + 1}{x^2 + x + 1} \Rightarrow k(x^2 + x + 1) = x^2 - x + 1$	**M1**	
	$\Rightarrow (k-1)x^2 + (k+1)x + (k-1) = 0$	**A1**	Answer Given
	$y = k$ a tangent iff $(k+1)^2 = 4(k-1)^2$ (May reason later on from $\Delta \geqslant 0$ for real points)	**M1**	A correct discriminant condition considered
	Then $k + 1 = 2(k-1) \Rightarrow k = 3$, $k + 1 = -2(k-1) \Rightarrow k = \frac{1}{3}$	**A1**	
	Convincingly reasoned from $\Delta = 0$ or $\Delta \geqslant 0$	**B1**	
	and the TPs are at $(-1, 3)$ and $(1, \frac{1}{3})$	**A1**	For both pairs Calculus methods score max. M1 A1 … M1 A1
7(b)		**B1**	Single curve through (0, 1)
		B1	Curve with range $[\frac{1}{3}, 3]$ – **FT (i)**'s y-values
		B1	$y = 1$ asymptote noted or clear from diagram
		B1	All correct

Question	Answer	Marks	Guidance
8(a)			
	Closed curve of generally correct shape	**B1**	Accept circle-ish
	Symmetry in *x*-axis	**B1**	
	(1, 0) and (–9, 0) and all correct	**B1**	Ignore TP/cusp at (1, 0) The 9 must be on-sketch or implied by clear scale
8(b)(i)	$Q = \left(5 + 4\sin\theta,\ \theta + \frac{1}{2}\pi\right)$	**B1**	Any one *r* correct
	$R = \left(5 + 4\cos\theta,\ \theta + \pi\right)$	**B1**	All three *r*'s correct
	$S = \left(5 - 4\sin\theta,\ \theta + \frac{3}{2}\pi\right)$	**B1**	Any one θ correct
	Note: – *Q*, *S* are interchangeable	**B1**	All three θ's correct (allow alternative equivalents modulo 2π in each case)
8(b)(ii)	$(OP)(OR) + (OQ)(OS) = (5 - 4c)(5 + 4c) + (5 + 4s)(5 - 4s)$ $= 25 - 16c^2 + 25 - 16s^2$ $= 50 - 16(c^2 + s^2)$	**M1**	Including multiplying out attempt
	$= 34$	**A1**	

Question	Answer	Marks	Guidance
9(a)	Expression $= (\alpha\beta - 1)(\beta\gamma - 1)(\gamma\alpha - 1)$	**M1**	Attempt to multiply out
	$= \alpha^2\beta^2\gamma^2 - \alpha\beta\gamma[\alpha+\beta+\gamma] + [\alpha\beta+\beta\gamma+\gamma\alpha] - 1$	**M1**	Setting up in terms of the symmetric expressions
	$= r^2 - (-r)(-p) + q - 1$	**A1**	Correct with α, β, γ replaced by p, q, r
	$= 0$ iff $r(r-p) = 1-q$		
	and so $\left(\alpha - \frac{1}{\beta}\right)\left(\beta - \frac{1}{\gamma}\right)\left(\gamma - \frac{1}{\alpha}\right) = 0$ iff $r(r-p) = 1-q$ i.e. one root is the reciprocal of another $\Leftrightarrow r(r-p) = 1-q$	**B1**	Be forgiving about deficient iff arguments
9(b)	$f(-r) = -r^3 + pr^2 - qr + r$	**M1**	
	$= -r(r^2 - rp + q - 1) = 0 \Rightarrow -r$ a root	**A1**	
	Alt. Roots $\alpha, \frac{1}{\alpha}, \beta$ **M1** $\Rightarrow -r = \alpha \times \frac{1}{\alpha} \times \beta = \beta$ **A1**		
9(c)	$x^3 + \frac{11}{12}x^2 - \frac{21}{4}x + 3 = 0 \Rightarrow p = -\frac{11}{12}, q = -\frac{21}{4}, r = -3$	**B1**	
	Testing: $r(r-p) = -3 \times -\frac{25}{12} = \frac{25}{4}$ and $1 - q = 1 + \frac{21}{4} = \frac{25}{4}$ so that $x = -3$ is a root	**B1**	If root -3 **or** factor $(x+3)$ stated, award both Bs
	$12x^3 + 11x^2 - 63x + 36 = (x+3)(12x^2 - 25x + 12)$ $= (x+3)(4x-3)(3x-4)$	**M1**	Full factorisation attempted
	leading to solutions $x = -3, \frac{3}{4}, \frac{4}{3}$	**A1**	Allow any sound method for solving the equation

Question	Answer	Marks	Guidance
10(a)	$\sin 3x = \text{Im}(\cos 3x + \text{i}\sin 3x) = \text{Im}(\text{c} + \text{is})^3$	M1	Seen or implied
	$= 3\text{c}^2\text{s} - \text{s}^3$	A1	Correct terms from binomial expansion
	$= 3(1 - \text{s}^2)\text{s} - \text{s}^3 = 3\text{s} - 4\text{s}^3$	A1	**Given Answer** legitimately obtained
	$\frac{1}{2}\left(\dfrac{1}{\sin x} - \dfrac{1}{\sin 3x}\right) \equiv \frac{1}{2}\left(\dfrac{\sin 3x - \sin x}{\sin x \sin 3x}\right)$	M1	
	$\equiv \frac{1}{2}\left(\dfrac{2\text{s} - 4\text{s}^3}{\text{s}\,.\sin 3x}\right)$	M1	Use of above (given) result
	$\equiv \dfrac{1 - 2\text{s}^2}{\sin 3x} \equiv \dfrac{\cos 2x}{\sin 3x}$	A1	**Given Answer** legitimately obtained
10(b)	$\displaystyle\sum_{r=0}^{n}\left(\frac{\cos[2\times 3^r x]}{\sin[3^{r+1}x]}\right) = \frac{1}{2}\sum_{r=0}^{n}\left(\frac{1}{\sin[3^r x]} - \frac{1}{\sin[3^{r+1}x]}\right)$	M1	Use of **(a)**'s result
	Minimum of 3 pairs used to illustrate	M1	Writing as either a sum of paired-terms **or** as the difference of two (essentially identical) series
	$= \frac{1}{2}\left(\dfrac{1}{\sin[3^0 x]} - \dfrac{1}{\sin[3^{n+1}x]}\right)$	M1	Using the *Difference Method* to cancel all intermediate terms
	$= \frac{1}{2}\left(\dfrac{1}{\sin x} - \dfrac{1}{\sin[3^{n+1}x]}\right)$	A1	

Question	Answer	Marks	Guidance
11(a)	$I_n = \int \cos^{n-1}\theta . \cos\theta \, d\theta$	M1	Correct splitting and attempt at integration by parts
	$= \cos^{n-1}\theta \sin\theta - \int (n-1)\cos^{n-2}\theta . -\sin\theta . \sin\theta \, d\theta$	A1	
	$= \cos^{n-1}\theta \sin\theta + (n-1)\int \cos^{n-2}\theta \, (1-\cos^2\theta) \, d\theta$	M1	
	$= \cos^{n-1}\theta \sin\theta + (n-1)\{I_{n-2} - I_n\}$	A1	All integrals in I_k form
	$n I_n = \sin\theta \cos^{n-1}\theta + (n-1)I_{n-2}$	A1	**Given Answer** legitimately obtained
11(b)(i)	$\frac{dx}{d\theta} = 2 + 2\cos 2\theta$ and $\frac{dy}{d\theta} = -2\sin 2\theta$	B1	Both correct
	$\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = 4 + 8\cos 2\theta + 4(\cos^2 2\theta + \sin^2 2\theta)$	M1	Attempted
	$= 8(1 + \cos 2\theta)$ or $16\cos^2\theta$	A1	Either form pre-integration
	$L = \int_0^{\frac{1}{2}\pi} (4\cos\theta) \, d\theta$	M1	Use of Arc-length integral with an integrable term
	$= [4\sin\theta]$	A1	Correctly integrated
	$= 4$	A1	**Given Answer** legitimately obtained

Question	Answer	Marks	Guidance
11(b)(ii)	$S = 2\pi\int(2\cos^2\theta)(4\cos\theta)\,\mathrm{d}\theta$	**M1**	Use of SA formula: allow $(1 + \cos2\theta)$ for y.
	$= 16\pi I_3$	**A1**	Not needed in I_k form.
	$I_1 = 1$ (e.g. from **(b)** above)	**B1**	
	Then 3 $I_3 = \left[\sin\theta\cos^2\theta\right]_0^{\frac{1}{2}\pi} + 2I_1 = 0 + 2$	**M1 A1**	Use of reduction formula
	Thus $I_3 = \frac{2}{3}$ and $S = \frac{32}{3}\pi$	**A1 A1**	
	Alt. to I_3 (final five marks) $\int\cos^3\theta\,\mathrm{d}\theta = \int(1-s^2)\,\mathrm{d}s$ using substn. $s = \sin\theta$ **M1 A1** $= s - \frac{1}{3}s^3$ **A1** Ignore limits here Then $I_3 = \frac{2}{3}$ from proper use of limits **A1** Must be clearly demonstrated and $S = \frac{32}{3}\pi$ **A1**		**Or** via $\int\cos^3\theta\,\mathrm{d}\theta$ $= \int\left(\frac{1}{4}\cos3\theta + \frac{3}{4}\cos\theta\right)\,\mathrm{d}\theta$

Question	Answer	Marks	Guidance
12(a)	$\sinh x \equiv 2\sinh\frac{1}{2}x\cosh\frac{1}{2}x \equiv 2\tanh\frac{1}{2}x \,.\, \dfrac{1}{\operatorname{sech}^2\frac{1}{2}x}$	**M1 M1**	Use of (at least) two relevant identities
	$\Rightarrow 1 = \dfrac{2t}{1-t^2} \Rightarrow 0 = t^2 + 2t - 1 \Rightarrow t = -1 \pm \sqrt{2}$	**M1 A1**	Creating and solving a quadratic for t
	$t > 0 \Rightarrow \tanh\frac{1}{2}x = \sqrt{2} - 1$	**B1**	Explanation of value (**Given Answer**)
	Alt. $\sinh x = \frac{1}{2}(e^x - e^{-x}) = 1 \Rightarrow e^x = \sqrt{2} + 1$ **M1 A1** $(e^x > 0)$ Then $\tanh\frac{1}{2}x = \dfrac{e^{\frac{1}{2}x} - e^{-\frac{1}{2}x}}{e^{\frac{1}{2}x} + e^{-\frac{1}{2}x}} = \dfrac{e^x - 1}{e^x + 1}$ **M1** $= \dfrac{\sqrt{2}}{\sqrt{2}+2} = \dfrac{1}{\sqrt{2}+1} \times \dfrac{\sqrt{2}-1}{\sqrt{2}-1} = \sqrt{2} - 1$ **M1 A1**		**Or** $\sinh^{-1}x = \ln\left(1+\sqrt{2}\right)$ from the Formula Book
12(b)	$\dfrac{d}{dx}\left(2\tan^{-1}(\tanh\frac{1}{2}x)\right) = 2 \times \dfrac{1}{1 + (\tanh\frac{1}{2}x)^2} \times \frac{1}{2}\operatorname{sech}^2\frac{1}{2}x$	**M1 A1**	$k \times \dfrac{1}{1 + (\tanh\frac{1}{2}x)^2} \times \ldots$ needed for M
	$= \dfrac{1}{\cosh^2\frac{1}{2}x + \sinh^2\frac{1}{2}x^2}$	**M1**	Use of suitable trig. identity
	$= \dfrac{1}{\cosh x}$	**A1**	**Given Answer**

Question	Answer	Marks	Guidance
12(c)	$\sec^2\theta\,\mathrm{d}\theta = 2\sinh x\cosh x\,\mathrm{d}x$	B1	
	Limits $(0, \frac{1}{4}\pi)$ for $\theta \to (0, \sinh^{-1}1)$ for x	B1	Note: $\sinh^{-1}1 = \ln\left(1+\sqrt{2}\right)$
	$I = \int \dfrac{\sinh x \,.\, 2\sinh x\cosh x}{\cosh^2 x}\,\mathrm{d}x$	M1	Full substitution attempted
	$= \int \dfrac{2\sinh^2 x}{\cosh x}\,\mathrm{d}x = 2\int \dfrac{\cosh^2 x - 1}{\cosh x}\,\mathrm{d}x$	M1	Integrable form attempted
	$= 2\int \left(\cosh x - \dfrac{1}{\cosh x}\right)\mathrm{d}x$	A1	
	$= 2\,(\sinh x - 2\tan^{-1}(\tanh\frac{1}{2}x))$	A1	Correct integration of an $a\cosh x + \dfrac{b}{\cosh x}$ expression
	$= 2 - 4.\frac{1}{8}\pi = 2 - \frac{1}{2}\pi$	A1	