**Cambridge International Examinations**
Cambridge Pre-U Certificate

**FURTHER MATHEMATICS** **1348/01**

Paper 1 Further Pure Mathematics **May/June 2016**

MARK SCHEME

Maximum Mark: 120

**Published**

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This document consists of **10** printed pages.

| Question | Answer | Marks | Notes |
|---|---|---|---|
| 1 | $\sum_{r=1}^{n}\left(8r^3+r\right)\equiv 8\sum_{r=1}^{n}r^3+\sum_{r=1}^{n}r$ | M1 | Splitting into separate series |
| | $\equiv 8\times\frac{1}{4}n^2(n+1)^2+\frac{1}{2}n(n+1)$ | M1<br>M1 | Both used<br>good factorisation attempt |
| | $\equiv\frac{1}{2}n(n+1)\left\{4n^2+4n+1\right\}$ | | |
| | $\equiv\frac{1}{2}n(n+1)(2n+1)^2$ | A1<br>[4] | Legitimate **(AG)** |
| 2 | $\begin{pmatrix}6\\2\\5\end{pmatrix}\times\begin{pmatrix}-6\\1\\4\end{pmatrix}=3\begin{pmatrix}1\\-18\\6\end{pmatrix}$ | M1<br>A1 | Attempt at vector products of the d.v.s (any suitable multiple) |
| | Shortest Distance = $\lvert(\mathbf{b}-\mathbf{a})\bullet\hat{\mathbf{n}}\rvert$ | M1 | |
| | $=\frac{1}{19}\begin{pmatrix}10\\-2\\5\end{pmatrix}\bullet\begin{pmatrix}1\\-18\\6\end{pmatrix}=\frac{1}{19}(10+36+30)$ | B1<br>B1 | $\lvert\hat{\mathbf{n}}\rvert$ correct<br>Sc. Prod. **ft** correct |
| | $=4$ | A1<br>[6] | |
| | Alternative method:<br><br>**M1 A1** for common normal $\mathbf{i}-18\mathbf{j}+6\mathbf{k}$<br>**M1 A1** for parallel planes $x-18y+6z=-55$ and $-131$<br>**M1 A1** for Sh.D formula, $\frac{\lvert 131-55\rvert}{\lvert\mathbf{n}\rvert}=\frac{76}{19}=4$ | | |
| 3 (i) | $\frac{2x^2-x-1}{2x-3}=k\Rightarrow 2x^2-(2k+1)x+(3k-1)=0$ | B1 | **(AG)** Shown legitimately |
| | For non-real $x$, $(2k+1)^2-8(3k-1)<0$ | M1 | Considering discriminant (or equivalent) |
| | $4k^2-20k+9<0\Rightarrow(2k-1)(2k-9)<0$ | M1 | Solving from $\Delta<0$ |
| | $\Rightarrow$ no curve for $\frac{1}{2}<k=y<\frac{9}{2}$ | A1<br>[4] | **(AG)** Must be satisfactorily explained |
| (ii) | TPs at $y=\frac{1}{2}$ $\quad y=\frac{9}{2}$ | M1 | First $y$ ($k$) substituted back |
| | i.e. $2x^2-2x+\frac{1}{2}=0$ $\quad 2x^2-10x+\frac{25}{2}=0$ | M1 | Second $y$ ($k$) substituted back |
| | $x=\frac{1}{2}$ $\quad x=\frac{5}{2}$ | A1A1<br>[4] | |
| | <u>Alternative method:</u><br>when $\Delta=0$, **M1** $x=$ "$-\frac{b}{2a}$" $=\frac{2k+1}{4}$<br>**M1** $\Rightarrow x=\frac{1}{2}$ $(y=\frac{1}{2})$ & $x=\frac{5}{2}$ $(y=\frac{9}{2})$ **A1 A1**<br>**Note:** For finding TPs via $\frac{dy}{dx}=0$, max. **M1 A1** since qn. asks for a "deduce" method | | |

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|---|---|---|---|
| 4 (i) | Attempt at det(**M**)<br>Det = 0 *Shown* | M1<br>A1<br>[2] | <br>(Or via full alternative algebraic method) |
| (ii) | $-x+3y+z=1$<br>$5x-y+2z=16$<br>$-x+y \quad =-2$<br>parametrisation attempt (or equivalent) started:<br>e.g. set $x=\lambda$, then $y=\lambda-2$<br>complete attempt: $z=1+\lambda-3\lambda+6=7-2\lambda$<br>all correct (p.v. and d.v.) … may be in vector<br>line eqn. form: $\mathbf{r}=\begin{pmatrix}0\\-2\\7\end{pmatrix}+\lambda\begin{pmatrix}1\\1\\-2\end{pmatrix}$<br><br><u>Alternative method 1:</u><br><br>**B1** as above, followed by (e.g.):<br>Finding two distinct points on the solution line; e.g. (2, 0, 3), (0, –2, 7) **M1 A1**<br>Then eqn. of line containing these 2 points **M1 A1** possibly **ft**<br>for line (of intersection) of 3 planes (given by the 3 eqns.) **B1**<br><br><u>Alternative method 2:</u><br><br>**B1** as above, followed by:<br>Vector product of any two plane normals **M1A1**<br>Finding coords. or p.v. of any pt. on line **B1**<br>Eqn. of line using these results appropriately **B1**<br>for line (of intersection) of 3 planes (given by the 3 eqns.) **B1** | <br>B1<br><br>M1<br><br>M1<br>A1A1<br>[6] | <br>for all three |
| 5 | Aux. Eqn. $m^2-4m+5=0$<br>$m=2\pm \mathrm{i}$<br>Comp. Fn. is $y_C=\mathrm{e}^{2x}(A\cos x+B\sin x)$<br>For Part. Intgl. try $y=y_p=a\,\mathrm{e}^{2x}$<br>Both $y'=2a\,\mathrm{e}^{2x}$ and $y''=4a\,\mathrm{e}^{2x}$<br>Subst^g. into given d.e. & solving to find $a$ :<br>$y_p=24\mathrm{e}^{2x}$<br>Gen. Soln. $y=\mathrm{e}^{2x}(A\cos x+B\sin x+24)$ | M1<br>A1<br>B1ft<br>B1<br>B1<br>M1<br>A1<br>B1ft<br>[8] | Including solving attempt<br><br><br><br><br>$(4a-8a+5a)\mathrm{e}^{2x}=24\mathrm{e}^{2x}$<br><br>$y_C+y_P$ provided $y_C$ has 2 arbitrary constants and $y_P$ has none.<br>Also, $A$, $B$ must be real here |

| Question | Answer | Marks | Notes |
|---|---|---|---|
| 6 (i) | For f($x$) = sinh $x$ + sin $x$ – 3$x$,<br>f(2.5) = – 0.851… < 0 and f(3) = 1.159… > 0<br>Change-of-sign (for a continuous fn.)<br>$\Rightarrow$ 2.5 < $\alpha$ < 3 | M1<br><br>A1<br>[2] | or LHS < RHS and then LHS > RHS<br><br>All correctly shown/explained |
| (ii) | $\sinh x + \sin x = \left(x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \frac{x^9}{9!} + \ldots\right) + \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \ldots\right)$<br>$= 2x + \frac{x^5}{60} + \ldots$<br>$2x + \frac{x^5}{60} = 3x \Rightarrow (x \neq 0)\ x^4 = 60$<br>$\Rightarrow \alpha \approx \sqrt[4]{60}$ (2.783 158 …) | M1<br><br>A1<br><br>B1<br>[3] | for use of both series (attempted)<br><br><br><br>**(AG)** shown legitimately |
| (iii) | Using $2x + \frac{x^5}{60} + \frac{x^9}{181\,440} = 3x$ with $x \neq 0$<br>Solving as a quadratic in $x^4$<br>$\alpha \approx 2.769\,8$ (to 4 d.p.)<br><br>[c.f. actual root 2.769 7 to 4 d.p.] | M1<br>M1<br>A1<br><br>[3] | <br>$x^8 + 3024x^4 - 181\,440 = 0$<br>from $x^4 = \sqrt{2\,467\,584} - 1512$,<br>$x = \sqrt[4]{58.854\,5\ldots}$ |
| 7 (i) | $\lvert z^3 \rvert = 2\sqrt{2}$ arg($z^3$) = $\frac{1}{4}\pi$<br>$\Rightarrow z = \left(\sqrt{2}, \frac{1}{12}\pi\right)$ cube-rooting modulus; arg ÷3<br>Other two roots: $\left(\sqrt{2}, \frac{3}{4}\pi\right)$ and $\left(\sqrt{2}, \frac{17}{12}\pi\right)$ | B1B1<br>M1M1<br>A1A1<br>[6] | <br>(in at least the first case) |
| (ii) | Equilateral Δ with vertices in approx. correct places<br>Area = $3 \times \frac{1}{2} \times \sqrt{2} \times \sqrt{2} \sin\left(\frac{2}{3}\pi\right) = \frac{3}{2}\sqrt{3}$<br>Accept **awrt** 2.60 (3 s.f.) from correct working | B1<br><br>M1A1<br><br>[3] | <br><br>Give M1 for any correct area |

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| 8 (i) (a) | (see table G below) | M1<br>A1<br>[2] | for mostly correct<br>for all correct |
| (b) | $(S, \times_{63})$ closed, since no new elements in table<br>$\times_{63}$ is associative (given)<br>1 is the identity element<br>Each (non-identity) element has a unique inverse:<br>$2 \leftrightarrow 32$, $4 \leftrightarrow 16$ and 8 is self-inverse | B1<br>B1<br>B1<br>[3] | All must be identified |
| (ii) (a) | (see table H below) | B1<br>B1<br>B1<br>B1<br>[4] | for last 3 elements (any forms)<br>for identity row/column (green)<br>for easy elements (gold) or $\geqslant$ 14 others<br>for all |
| (b) | Proper subgroups of $H$ are (condone inclusion of $\{e\}$ and $H$):<br>$\{e, x\}$, $\{e, xy\}$, $\{e, yx\}$ and $\{e, y, y^2\}$ | B1B1<br>[2] | B1 Any 2; +B1 all 4 and no extras |
| (c) | $G$ and $H$ are NOT isomorphic<br>e.g. Different numbers of self-inverse elements / elements of order 3<br>or $G$ cyclic, $H$ non-cyclic or $G$ abelian, $H$ non-abelian | B1<br>[1] | Correct conclusion WITH a valid reason |

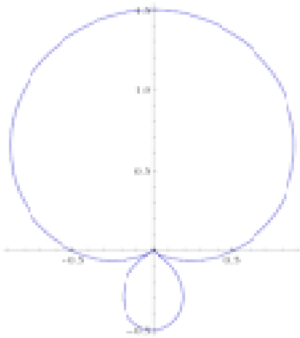
Table for 8 (i) (a):

| **G** | **1** | **2** | **4** | **8** | **16** | **32** |
|---|---|---|---|---|---|---|
| **1** | 1 | 2 | 4 | 8 | 16 | 32 |
| **2** | 2 | 4 | 8 | 16 | 32 | 1 |
| **4** | 4 | 8 | 16 | 32 | 1 | 2 |
| **8** | 8 | 16 | 32 | 1 | 2 | 4 |
| **16** | 16 | 32 | 1 | 2 | 4 | 8 |
| **32** | 32 | 1 | 2 | 4 | 8 | 16 |

Table for 8 (ii) (a):

| $H$ | $e$ | $x$ | $y$ | $y^2$ | $xy$ | $yx$ |
|---|---|---|---|---|---|---|
| $e$ | $e$ | $x$ | $y$ | $y^2$ | $xy$ | $yx$ |
| $x$ | $x$ | $e$ | $xy$ | $yx$ | $y$ | $y^2$ |
| $y$ | $y$ | $yx$ | $y^2$ | $e$ | $x$ | $xy$ |
| $y^2$ | $y^2$ | $xy$ | $e$ | $y$ | $yx$ | $x$ |
| $xy$ | $xy$ | $y^2$ | $yx$ | $x$ | $e$ | $y$ |
| $yx$ | $yx$ | $y$ | $x$ | $xy$ | $y^2$ | $e$ |

| Question | Answer | Marks | Notes |
|---|---|---|---|
| 9 (i) | $\alpha+\beta+\gamma=a$, $\alpha\beta+\beta\gamma+\gamma\alpha=b$ and $\alpha\beta\gamma=c$ | B1B1<br>[2] | B1 any 2 correct; + B1 all 3 correct |
| (ii) | $\alpha^2+\beta^2+\gamma^2=(\alpha+\beta+\gamma)^2-2(\alpha\beta+\beta\gamma+\gamma\alpha)$ | M1 | |
| | $=a^2-2b$ | A1 | |
| | $\alpha^2\beta^2+\beta^2\gamma^2+\gamma^2\alpha^2=(\alpha\beta+\beta\gamma+\gamma\alpha)^2-2\alpha\beta\gamma(\alpha+\beta+\gamma)$ | M1 | |
| | $=b^2-2ac$ | A1<br>[4] | |
| (iii) | $(\alpha-2\beta\gamma)(\beta-2\gamma\alpha)(\gamma-2\alpha\beta)$ | | |
| | $=\left(\alpha\beta-2\beta^2\gamma-2\alpha^2\gamma+4\gamma^2\alpha\beta\right)(\gamma-2\alpha\beta)$ | M1 | |
| | $=\alpha\beta\gamma-2(\alpha^2\beta^2+\beta^2\gamma^2+\gamma^2\alpha^2)+4\alpha\beta\gamma(\alpha^2+\beta^2+\gamma^2)-8(\alpha\beta\gamma)^2$ | M1 | Collecting up in terms of the symmetric fns. |
| | $=c-2\left(b^2-2ac\right)+4c\left(a^2-2b\right)-8c^2$ | M1 | Use of **(i)**'s and **(ii)**'s results |
| | $=c\left(1+4a+4a^2\right)-2\left(b^2+4bc+4c^2\right)$ | | |
| | $=c(2a+1)^2-2(b+2c)^2$ | A1<br>[4] | **legitimately** |
| | <u>Alternative method:</u> | | |
| | Using $\alpha\beta\gamma=c$,<br>$(\alpha-2\beta\gamma)(\beta-2\gamma\alpha)(\gamma-2\alpha\beta)$<br>$=\left(\alpha-\frac{2c}{\alpha}\right)\left(\beta-\frac{2c}{\beta}\right)\left(\gamma-\frac{2c}{\gamma}\right)$<br>$=\frac{1}{\alpha\beta\gamma}\left(\alpha^2-2c\right)\left(\beta^2-2c\right)\left(\gamma^2-2c\right)=$<br>$\frac{1}{c}\left((\alpha\beta\gamma)^2-2c\sum\alpha^2\beta^2+4c^2\sum\alpha^2-8c^3\right)$<br>$=\frac{1}{c}\left(c^2-2c\left[b^2-2ac\right]+4c^2\left[a^2-2b\right]-8c^3\right)$<br>= etc. as above | | |
| (iv) | One root is the product of the other two<br>$\Leftrightarrow$ $(\alpha-2\beta\gamma)(\beta-2\gamma\alpha)(\gamma-2\alpha\beta)=0$ | | |
| | $\Leftrightarrow$ $c(2a+1)^2=2(b+2c)^2$ | B1 | **legitimately** |
| | Must reason $\Rightarrow$ and $\Leftarrow$ explicitly (or together) | [1] | |

| Question | Answer | Marks | Notes |
|---|---|---|---|
| 10 (i) |  | M1A1 | $\frac{1}{2}+\sin\theta=0$ when $\theta=\frac{7}{6}\pi$, $\frac{11}{6}\pi$ |
| | | B1 | Symmetry in $y$-axis |
| | | B1 | $\left(\frac{1}{2},\ 0\right)$ on initial line |
| | | B1 | Correct upper portion |
| | | B1 [6] | Correct lower portion |
| (ii) | $A=\left(\frac{1}{2}\right)\int_0^{2\pi}\left(\frac{1}{2}+\sin\theta\right)^2\ \mathrm{d}\theta$ | M1 | Penalise incorrect multiples with final A0 |
| | $=\frac{1}{2}\int_0^{2\pi}\left(\frac{1}{4}+\sin\theta+\frac{1}{2}-\frac{1}{2}\cos 2\theta\right)\mathrm{d}\theta$ | M1 | Double-angle formula |
| | $=\frac{1}{2}\left[\frac{3}{4}\theta-\cos\theta-\frac{1}{4}\sin 2\theta\right]_0^{2\pi}$ | A1 | correctly integrated 3 suitable terms |
| | $=\frac{3}{4}\pi$ | A1 [4] | |

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|---|---|---|---|
| 11 (i) | $F_3 = 2,\ F_4 = 3,\ \ F_5 = 5,\ \ F_6 = 8$ | B1 [1] | all |
| (ii) (a) | $p_2(x) = 1 + \frac{1}{x+1} = \frac{x+2}{x+1}$ | B1 | |
| | $p_3(x) = \frac{2x+3}{x+2}$ | B1 | |
| | $p_4(x) = \frac{3x+5}{2x+3}$ | B1 [3] | **(AG))** |
| (b) | $p_n(x) = \frac{F_n\ x + F_{n+1}}{F_{n-1}\ x + F_n}$ | B1 | |
| | Result is true for $n = 2$ (and 3 and 4) | B1 | May be mentioned in later in their "round up" |
| | Assuming $p_k(x) = \frac{F_k\ x + F_{k+1}}{F_{k-1}\ x + F_k}$ (not separate from their conjecture) | | |
| | $p_{k+1}(x) = 1 + \frac{F_{k-1}\ x + F_k}{F_k\ x + F_{k+1}}$ | M1 | |
| | $= \frac{F_k\ x + F_{k+1}}{F_k\ x + F_{k+1}} + \frac{F_{k-1}\ x + F_k}{F_k\ x + F_{k+1}}$ | | |
| | $= \frac{(F_k + F_{k-1})\ x + (F_k + F_{k+1})}{F_k\ x + F_{k+1}}$ | M1 | Collecting coeffts. into successive Fib. terms |
| | $= \frac{F_{k+1}\ x + F_{k+2}}{F_k\ x + F_{k+1}}$ | A1 | |
| | which is the required formula with $n = k + 1$. Accept this as sufficient that proof follows by induction. | [5] | |

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|---|---|---|---|
| 12 (i) | $y = \ln\left(\tanh\frac{1}{2}x\right) \Rightarrow \frac{dy}{dx} = \frac{1}{\tanh\frac{1}{2}x} \cdot \frac{1}{2}\text{sech}^2\frac{1}{2}x$ | M1A1 | |
| | $= \text{cosech}\, x$ | A1 [3] | **(AG)** |
| (ii) (a) | $L_n = \int_n^{2n} \sqrt{1+\text{cosech}^2 x}\, dx$ | M1 | |
| | $= \int_n^{2n} \coth x\, dx$ | A1 | |
| | $= [\ln(\sinh x)]$ | A1 | correct integrn. |
| | $\ln\left(\frac{\sinh 2n}{\sinh n}\right) = \ln\left(\frac{e^{2n} - e^{-2n}}{e^n - e^{-n}}\right)$ | M1 | |
| | $\approx \ln\left(\frac{e^{2n}}{e^n}\right)$, for large $n$, $= \ln\left(e^n\right) = n$ | A1 [5] | **legitimately** |
| | **OR**<br>$\ln\left(\frac{\sinh 2n}{\sinh n}\right) = \ln\left(2\cosh n\right) = \ln\left(e^n + e^{-n}\right)$ **M1** | | |
| | $\approx \ln\left(e^n\right)$ for large $n$, $= n$ **A1** | | **legitimately** |
| (b) | Method (sketch or statement) to indicate that $C$ asymptotically "merges" with the $x$-axis | M1 | |
| | so that $C$ is approximately a horizontal straight-line from $(n, 0)$ to $(2n, 0)$ | A1 [2] | |
| 13 (i) (a) | Let $y = \sec^{-1}x$, i.e. $\sec y = x$ | | |
| | $\Rightarrow \cos y = \frac{1}{x} \Rightarrow y = \cos^{-1}\left(\frac{1}{x}\right)$ | B1 | |
| | Then $\frac{d}{dx}\left(\sec^{-1}x\right) = \frac{d}{dx}\left(\cos^{-1}\frac{1}{x}\right)$ | | |
| | $= -\frac{1}{\sqrt{1-(1/x)^2}} \times \frac{-1}{x^2}$ | M1 | (Using MF20 and the *Chain Rule*) |
| | $= \frac{1}{x\sqrt{x^2-1}}$ | A1 [3] | **(AG)** |
| | [Allow **M1 A1** for valid non-"deduced" approaches] | | |

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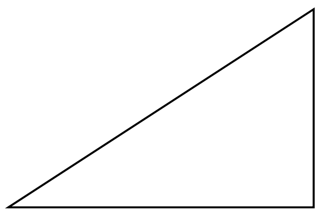
| (b) | $\int \sec^{-1} x \,.\, 1 \ \mathrm{d}x$ | M1 | Use of integration by "parts" |
| | $= x \,.\, \sec^{-1} x - \int x \,.\, \frac{1}{x\sqrt{x^2-1}} \mathrm{d}x$ | A1 A1 | |
| | $= \left[ x \,.\, \sec^{-1} x - \cosh^{-1} x \right]$ | A1 [4] | Condone lack of "+ $C$" |
| (ii) (a) | $\frac{1}{x\sqrt{x^2-1}} = \frac{1}{\sqrt{2}} \Rightarrow x^2\left(x^2-1\right) = 2$ | | |
| | $\Rightarrow x^4 - x^2 - 2 = \left(x^2-2\right)\left(x^2+1\right) = 0$ | M1 | |
| | $\Rightarrow x = \sqrt{2}$ and $y = \frac{1}{4}\pi$ | A1 A1 | i.e. $P = (\sqrt{2}, \frac{1}{4}\pi)$ |
| |   | | |
| | $\frac{\frac{1}{4}\pi}{\sqrt{2}-c} = \frac{1}{\sqrt{2}}$ | M1 A1 | or by $y - \frac{1}{4}\pi = \frac{1}{\sqrt{2}}\left(x - \sqrt{2}\right)$ & $y = 0$ |
| | $c = \sqrt{2} - \frac{\pi\sqrt{2}}{4}$ | A1 [6] | i.e. $Q = \left(\sqrt{2} - \frac{\pi\sqrt{2}}{4},\ 0\right)$ |
| (b) | Area $\Delta = \frac{1}{2} \times \frac{\pi\sqrt{2}}{4} \times \frac{\pi}{4} = \frac{\pi^2\sqrt{2}}{32}$ | B1 | |
| | Area under curve = $\sqrt{2} \,.\, \frac{\pi}{4} - \ln\left(1+\sqrt{2}\right)$ | B1 | using **(iii)**'s answer and the limits $(1, \sqrt{2})$ |
| | Then $R = \frac{\pi^2\sqrt{2}}{32} - \frac{\pi\sqrt{2}}{4} + \ln\left(1+\sqrt{2}\right)$ | M1 | Difference in areas |
| | $= \ln\left(1+\sqrt{2}\right) - \frac{\pi(8-\pi)\sqrt{2}}{32}$ | A1 [4] | **(AG)** |