UNIVERSITY OF CAMBRIDGE INTERNATIONAL EXAMINATIONS
Cambridge International Level 3 Pre-U Certificate
Principal Subject

FURTHER MATHEMATICS **9795/01**

Paper 1 Further Pure Mathematics **May/June 2013**

3 hours

Additional Materials: Answer Booklet/Paper
Graph Paper
List of Formulae (MF20)

* 1 1 2 6 4 9 0 4 1 3 *

READ THESE INSTRUCTIONS FIRST

If you have been given an Answer Booklet, follow the instructions on the front cover of the Booklet.
Write your Centre number, candidate number and name on all the work you hand in.
Write in dark blue or black pen.
You may use a soft pencil for any diagrams or graphs.
Do not use staples, paper clips, highlighters, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
The use of an electronic calculator is expected, where appropriate.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.
The number of marks is given in brackets [ ] at the end of each question or part question.
The total number of marks for this paper is 120.

This document consists of **4** printed pages.

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[Turn over

1 By completing the square, or otherwise, find the exact value of $\displaystyle\int_2^6 \frac{1}{x^2 - 6x + 12}\,dx$. [4]

2 Use the standard Maclaurin series expansions given in the List of Formulae MF20 to show that

$$\tfrac{1}{2}\ln\left(\frac{1+x}{1-x}\right) \equiv \tanh^{-1} x \quad \text{for } -1 < x < 1. \qquad [4]$$

3 The curve C has equation $y = \dfrac{x+1}{x^2-4}$.

(i) Show that the gradient of C is always negative. [3]

(ii) Sketch C, showing all significant features. [6]

4 **(i)** Find a vector which is perpendicular to both of the vectors

$$\mathbf{d}_1 = \mathbf{i} + 2\mathbf{j} + 4\mathbf{k} \quad \text{and} \quad \mathbf{d}_2 = 9\mathbf{i} - 3\mathbf{j} + \mathbf{k}. \qquad [2]$$

(ii) Determine the shortest distance between the skew lines with equations

$$\mathbf{r} = 2\mathbf{i} + 4\mathbf{j} + 3\mathbf{k} + \lambda(\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}) \quad \text{and} \quad \mathbf{r} = \mathbf{i} + \mathbf{j} + 10\mathbf{k} + \mu(9\mathbf{i} - 3\mathbf{j} + \mathbf{k}). \qquad [5]$$

5 Let $z = \cos\theta + \mathrm{i}\sin\theta$.

(i) Prove the result $z^n - \dfrac{1}{z^n} = 2\mathrm{i}\sin n\theta$. [2]

(ii) Use this result to express $\sin^5\theta$ in the form $A\sin 5\theta + B\sin 3\theta + C\sin\theta$, for constants A, B and C to be determined. [5]

6 The curve P has polar equation $r = \dfrac{1}{1-\sin\theta}$ for $0 \leqslant \theta < 2\pi$, $\theta \neq \frac{1}{2}\pi$.

(i) Determine, in the form $y = \mathrm{f}(x)$, the cartesian equation of P. [3]

(ii) Sketch P. [2]

(iii) Evaluate $\displaystyle\int_\pi^{2\pi} \frac{1}{(1-\sin\theta)^2}\,d\theta$. [3]

7 **(i)** Express $x^3 + y^3$ in terms of $(x+y)$ and xy. [2]

(ii) The equation $t^2 - 3t + \frac{8}{9} = 0$ has roots α and β.

(a) Determine the value of $\alpha^3 + \beta^3$. [2]

(b) Hence express 19 as the sum of the cubes of two positive rational numbers. [3]

8 Let $G = \{g_1, g_2, g_3, \ldots, g_n\}$ be a finite abelian group of order n under a multiplicative binary operation, where $g_1 = e$ is the identity of G.

(i) Let $x \in G$. Justify the following statements:

(a) $xg_i = xg_j \iff g_i = g_j$; [2]

(b) $\{xg_1, xg_2, xg_3, \ldots, xg_n\} = G$. [1]

(ii) By considering the product of all G's elements, and using the result of part **(i)(b)**, prove that $x^n = e$ for each $x \in G$. [3]

(iii) Explain why

(a) this does not imply that all elements of G have order n, [1]

(b) this argument cannot be used to justify the same result for non-abelian groups. [1]

9 The plane transformation T is the composition (in this order) of

- a reflection in the line $y = x\tan\frac{1}{8}\pi$; followed by
- a shear parallel to the y-axis, mapping $(1, 0)$ to $(1, 2)$; followed by
- a clockwise rotation through $\frac{1}{4}\pi$ radians about the origin; followed by
- a shear parallel to the x-axis, mapping $(0, 1)$ to $(-2, 1)$.

Determine the matrix $\mathbf{M}$ which represents T, and hence give a full geometrical description of T as a single plane transformation. [8]

10 **(a)** Given that $y = kx\cos x$ is a particular integral for the differential equation

$$\frac{d^2y}{dx^2} + y = 4\sin x,$$

determine the value of k and find the general solution of this differential equation. [8]

(b) The variables x and y satisfy the differential equation

$$\frac{d^2y}{dx^2} + y^2\frac{dy}{dx} + xy = 5x - 19.$$

(i) Given that $y = 2$ and $\frac{dy}{dx} = 1$ when $x = 1$, find the value of $\frac{d^3y}{dx^3}$ when $x = 1$. [6]

(ii) Deduce the Taylor series expansion for y in ascending powers of $(x - 1)$, up to and including the term in $(x - 1)^3$, and use this series to find an approximation correct to 3 decimal places for the value of y when $x = 1.1$. [4]

11 **(i)** Determine p and q given that $(p + iq)^2 = 63 - 16i$ and that p and q are real. [4]

(ii) Let $f(z) = z^3 - Az^2 + Bz - C$ for complex numbers A, B and C.

(a) Given that the cubic equation $f(z) = 0$ has roots $\alpha = -7i$, $\beta = 3i$ and $\gamma = 4$, determine each of A, B and C. [4]

(b) Find the roots of the equation $f'(z) = 0$. [5]

12 Given $y = x\mathrm{e}^{2x}$,

(i) find the first four derivatives of y with respect to x, [4]

(ii) conjecture an expression for $\dfrac{\mathrm{d}^n y}{\mathrm{d}x^n}$ in the form $(ax + b)\mathrm{e}^{2x}$, where a and b are functions of n, [2]

(iii) prove by induction that your result holds for all positive integers n. [5]

13 **(i)** Use the definitions $\tanh\theta = \dfrac{\mathrm{e}^{\theta} - \mathrm{e}^{-\theta}}{\mathrm{e}^{\theta} + \mathrm{e}^{-\theta}}$ and $\operatorname{sech}\theta = \dfrac{2}{\mathrm{e}^{\theta} + \mathrm{e}^{-\theta}}$ to prove the results

(a) $\tanh^2\theta \equiv 1 - \operatorname{sech}^2\theta$,

(b) $\dfrac{\mathrm{d}}{\mathrm{d}\theta}(\tanh\theta) = \operatorname{sech}^2\theta$. [4]

(ii) Let $I_n = \displaystyle\int_0^{\alpha} \tanh^{2n}\theta\,\mathrm{d}\theta$ for $n \geqslant 0$, where $\alpha > 0$.

(a) Show that $I_{n-1} - I_n = \dfrac{\tanh^{2n-1}\alpha}{2n - 1}$ for $n \geqslant 1$. [4]

Given that $\alpha = \frac{1}{2}\ln 3$,

(b) evaluate I_0, [1]

(c) use the method of differences to show that $I_n = \frac{1}{2}\ln 3 - \displaystyle\sum_{r=1}^{n} \frac{\left(\frac{1}{2}\right)^{2r-1}}{2r - 1}$ and deduce the sum of the infinite series $\displaystyle\sum_{r=0}^{\infty} \frac{1}{(2r + 1)4^r}$. [7]